AN ANALYTIC SOLAR MODEL

PHYSICAL PRINCIPLES AND MATHEMATICAL STRUCTURE

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Abstract

Physical principles and mathematical structure involved in deriving an analytical representation of the internal structure of the Sun is discussed. For a two-parameter family of a non-linear matter density distribution, the run of mass, pressure, temperature, and luminosity throughout the Sun is presented in terms of Gauss' hypergeometric function. The system of differential equations governing hydrostatic equilibrium and energy conservation for the spherical Sun is proved to be a laboratory for the application of special functions.

1   Introduction

The structure of the Sun is determined by conditions of mass conservation, momentum conservation, energy conservation, and specific modes of energy transport through the Sun. For considerations of its internal structure,

rotation and magnetic fields can be neglected so that the Sun is spherically symmetric. It may come as a surprise that much of what has been observed and theorized about the Sun can be accounted for in terms of very basic physical laws: Newton's laws of gravity and of motion, the first two laws of thermodynamics, Einstein's law of the equivalence of mass and energy, Boyle's law and Charles' law of perfect gases, and Heisenberg's uncertainty principle. The outline of the theory of the structure and evolution of the Sun has clearly belonged to the first half of the twentieth century and is connected with names like Lane, Emden, Schwarzschild, Eddington, Chandrasekhar, Hoyle, and Fowler (a brief outline of this history is contained in Mathai and Haubold, 1988). In the second half of this century, observation and theory of the Sun were greatly refined, partly by new observationed techniques and partly by computer simulations of its structure and evolution. By and large, the picture of the structure and evolution of the Sun seems to be well understood. However, there is a serious discrepancy between the theory of how the Sun shines and the most direct experimental test of this theory. This discrepancy is more specifically called 'the solar neutrino problem' and refers to the fact that the Sun is a volume source of neutrinos, particles produced by thermonuclear reactions in the deep interior of the Sun, and that the copious flux of solar neutrinos predicted by theory does not match the flux detected by experiment on Earth over the past 25 years, connected with names like Davis and Bahcall (Bahcall, 1989). Solar neutrinos are the only particles that have the ability to penetrate from the center of the

Sun to the surface almost without interaction with solar matter, escaping freely into space carrying the most direct information about physical processes in the deep solar interior. The solar neutrino problem, which shall not be the subject of this paper, stimulated further studies of solar models, both by employing computing machines and using analytical techniques of mathematics. This was also the justification of an effort to reconsider the derivation of analytic solutions to the system of differential equations of solar structure based on the very basic physical laws referenced above. Particularly, the solar neutrino problem has been considered to be a reason to pursue more actively obtaining analytic formulae yielding a description of the gravitationally stabilized solar fusion reactor and showing that methods of the integration theory of generalized special functions applied to solar physics constitute a laboratory for the application of these functions (Mathai and Haubold, 1988).

The nuclear reactions which cause the Sun to evolve are sufficiently slow that the Sun may be assumed to pass through a series of equilibrium configurations. The model for the internal structure of the current Sun may be thought of as representing the Sun at an instant of time. Separating the time dependence of the evolution of the Sun from the equations governing its internal structure allow to replace the time dependent partial differential equations by four simultaneous, non-linear, ordinary differential equations of the first order. The four equations represent the radial gradients of mass, $M(r)$, pressure, $P(r)$, temperature $T(r)$, and luminosity, $L(r)$. Since there are four equations but more than

four unknown physical variables, one needs additional constitutive equations, before the system can be solved: an equation of state for solar matter, a nuclear energy production rate, and an opacity law. The full system of equations must be solved subject to at most four boundary conditions at the surface and the centre of the Sun. These boundary conditions ensure that the structure of the Sun can be calculated from the four differential equations but they do not ensure that there is a single unique solution (Chandrasekhar, 1939; Stein, 1966). The procedure of numerically integrating the solar structure equations takes advantage of large electronic computers and makes it possible to include a variety of detailed physical effects and to vary parameters at will (Bahcall, 1989; Noels et al., 1993). A second procedure to provide a solar model, but whose contents can be understood intuitively not resorting to numerical techniques, starts with Buckingham's theorem that a system characterized by n physical variables can be described by an ensemble of n-r dimensionless products of variables, where r is the number of variables whose dimensionless representations are linearly independent. This approach provides a qualitative explanation of the fundamental stellar structure equations through dimensional analysis. This analysis suggests that more detailed physical and mathematical theories are essentially theories of factors of proportionality. They eventually yield numerical values for these proportional factors because more physical assumptions have to be made (Bhaskar and Nigam, 1991). The third procedure treats the solar structure equations by rigorous mathematics leading to the Lane-Emden equation which is a second-

order non-linear differential equation describing the structure of a polytrope gas sphere. However, explicit analytic solutions of the Lane-Emden equation exist only for values n=0,1, and 5 of the polytrope of index n, not covering the specific physical model for the internal structure of the Sun (Chandrasekhar, 1939; Horedt, 1990).

Additional to the three procedures of constructing solar models referred to above, there is an approach to find solutions of the solar structure equations by making a specific assumption for straightforward analytical integration of these equations. It is possible to obtain analytic solar models by separating the hydrostatic component from the energy-transport component of the structure equations. For that purpose, an analytic density distribution, namely, that the matter density in the Sun varies non-linearly from the center to the thought surface, where the density goes to zero, must be assumed (Stein, 1966; Mathai and Haubold, 1988). Then it is possible to integrate the equations of mass conservation, hydrostatic equilibrium, and energy conservation through the Sun. Together with the equation of state of a perfect gas, the run of density, $\rho(r)$, mass, $M(r)$, pressure, $P(r)$, temperature, $T(r)$, and luminosity, $L(r)$, is determined and can be derived in the form of analytic formulae. The physics of the problem requires only three independent boundary conditions: $M(r) \to 0$ and $L(r) \to 0$ at radial distance $r = 0$; $T \to T_0 = 0$ and $\rho \to \rho_0 = 0$ at the radius $r = R_\odot$ of the gaseous configuration. The requirement that ρ and T tend simultaneously to specific values, in this case zero, is only one condition since

the point at which this occurs is arbitrary. This ambiquity can be removed, in principle, by assigning the total mass. The boundary is required to be at the point where $M(r = R_\odot) = M_\odot$ and this provides the fourth condition. Hence, the central density, pressure, temperature, and total rate of energy generation are determined as a function of the Sun's mass and radius. However, by assuming an analytic matter density distribution, the energy-transport equation of the system of structure differential equations can be satisfied at only one typical point in the Sun. The procedure thus established to construct an analytic model of the solar interior allows to determine the factors of proportionality which remain to be an open problem in the dimensional analysis. The procedure also reveales that the run of all physical variables for the solar model can be expressed in terms of Gauss' hypergeometric function. (Luke, 1969; Mathai, 1993)

2 Matter Density Distribution

For the integration of the system of differential equations governing the internal structure of the Sun one has to make a choice for an unknown function that still leaves room for physical justification of this choice. By intuition one expects that the mass is an increasing function while density, pressure and temperature are decreasing functions throughout the Sun towards its surface. Thus we make a working hypothesis that the matter density distribution $\rho(r)$ varies

with the distance variable r,

$$(2.1) \qquad \rho(r) = \rho_c \left[1 - \left(\frac{r}{R_\odot} \right)^\delta \right]^\gamma, \delta > 0, \gamma > 0, 0 \leq \frac{r}{R_\odot} \leq 1,$$

where δ and γ are kept as free parameters to satisfy at a later point that the density distribution determines properly the mass, pressure, and temperature distribution in the interior of the Sun. Equation (2.1) takes into account that the chosen density distribution reflects the central value of the density $\rho(r = 0) = \rho_c$ and satisfies the boundary conditon $\rho(r = R_\odot) = 0$, where $R_\odot$ denotes the solar radius. Also, equation (2.1) implies that $\rho \propto M/R^3$, where the constant of proportionality depends only on the radial mass distribution and the radial distance.

3 Distribution of Mass

If $M(r)$ represents the total mass contained within the radius r, and $\rho(r)$ is the density at r, then

$$(3.1) \qquad \frac{dM(r)}{dr} = 4\pi r^2 \rho(r).$$

Using the assumed non-linear density distribution in equation (2.1), integration of (3.1) throughout the Sun leads to the distribution of mass

$$(3.2) \qquad M\left(\frac{r}{R_\odot} \right) = \frac{4\pi}{3} \rho_c R_\odot^3 \left(\frac{r}{R_\odot} \right)^3 {}_2F_1\left(-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; \left(\frac{r}{R_\odot} \right)^\delta \right),$$

where ${}_2F_1(.)$ denotes Gauss' hypergeometric function, containing the parameters δ and γ of the matter density distribution in equation (2.1) (Luke, 1969; Mathai,

1993). Equation (3.2) satisfies the boundary condition $M(r = 0) = 0$ and can be used to determine the central value of the matter density distribution ρ_c in equation (2.1) in terms of the parameters δ and γ of the chosen model of the Sun. The condition $M(r = R_\odot) = M_\odot$ in equation (3.2) reveals that

$$(3.3) \qquad \rho_c = \frac{3M_\odot}{4\pi R_\odot^3} \frac{(\frac{3}{\delta} + 1)(\frac{3}{\delta} + 2) \cdots (\frac{3}{\delta} + \gamma)}{\gamma!},$$

if γ in equation (2.1) is kept as a positive integer and subsequently using the specific relation for Gauss' hypergeometric function of argument one (Luke, 1969; Mathai, 1993), that is

$$_2F_1(a, b; c; 1) = \Gamma(c)\Gamma(c - a - b)/\Gamma(c - a)\Gamma(c - b).$$

Equation (3.3) can be used to select the appropriate values of the parameters δ and γ specifying the solar model with matter density distribution in equation (2.1).

4 Distribution of Pressure

If $g = GM(r)/r^2$ is the gravitational force per unit mass at r due to the attraction of the mass interior to r, then

$$(4.1) \qquad \frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

is the equation of hydrostatic equilibrium of the spherical self- gravitating Sun with $dP(r)/dr$ being the pressure gradient. The internal pressure produced by the weight of the overlying layers increases towards the centre while the

gas pressure must increase correspondingly to achieve the balance of forces for equilibrium. This increase is obtained by inward increases of both temperature and density. The internal pressure needed to balance is the gravitational force per unit mass (GM/R^2) times the mass per unit area (M/R^2) which gives that $P \propto GM^2/R^4$ for any spherical body in hydrostatic self- gravitation, where the constant of proportionality is again determined by the radial distribution of mass in the Sun and the particular radial distance at which P is measured. The constant of proportionality can be determined by integrating equation (4.1) throughout the Sun by using equations (2.1) and (3.2) for the density and mass distribution, respectively. We obtain

$$
\begin{aligned}
P(\frac{r}{R_\odot}) \;=\; & \frac{9}{4\pi} G \frac{M_\odot^2}{R_\odot^4} \left[\frac{(\frac{3}{\delta}+1)(\frac{3}{\delta}+2)\ldots(\frac{3}{\delta}+\gamma)}{\gamma!} \right]^2 \\
& \times \frac{1}{\delta^2} \sum_{m=0}^{\infty} \frac{(-\gamma)_m}{m!(\frac{3}{\delta}+m)(\frac{2}{\delta}+m)} \left[\frac{\gamma!}{(\frac{2}{\delta}+m+1)_\gamma} \right. \\
& \left. - \left(\frac{r}{R_\odot}\right)^{m\delta+2} {}_2F_1\left(-\gamma, \frac{2}{\delta}+m; \frac{2}{\delta}+m+1; \left(\frac{r}{R_\odot}\right)^\delta \right) \right],
\end{aligned}
$$

(4.2)

where ${}_2F_1(.)$ is Gauss' hypergeometric function and $(-\gamma)_m = \Gamma(-\gamma+m)/\Gamma(-\gamma)$ is Pochhammer's symbol that often appears in series expansions for hypergeometric functions. The solution of equation (4.1) given in equation (4.2) confirms the condition $P(r = R_\odot) = 0$ and gives the central value of the pressure according to the chosen solar model characterized by δ and γ in equation (2.1):

$$P_c = \frac{9}{4\pi} G \frac{M_\odot^2}{R_\odot^4} \left[\frac{(\frac{3}{\delta}+1)(\frac{3}{\delta}+2)\ldots(\frac{3}{\delta}+\gamma)}{\gamma!} \right]^2$$

$$\times \frac{1}{\delta} \sum_{m=0}^{\infty} \frac{(-\gamma)_m \gamma!}{m!(\frac{3}{\delta}+m)(\frac{2}{\delta}+m)(\frac{3}{\delta}+m+1)_\gamma}.$$

(4.3)

5 Temperature Distribution

The simplest theory of solar structure is that of a polytrope. These solar models obey an equation of state of the form $P = K\rho^{(n+1)/n}$ throughout the gas sphere. Since temperature does not explicitly occur in this relation between ρ and P, equations (3.1) and (4.1) may be solved independently of the temperature and luminosity gradients. This equation of state leads to the Lane-Emden equation for polytropic gas spheres, which is an ordinary differential equation of second order, but can be reduced, by suitable transformations of the variables, to an equation of the first order (Chandrasekhar, 1939; Horedt, 1990). In the Sun the density is so low that at the temperatures involved the solar material behaves almost as a perfect gas, having molecular weight μ, obeying the perfect gas law

(5.1)
$$P = \frac{kN_A}{\mu} \rho T,$$

where k is Boltzmann's constant and N_A Avogadro's number. Substituting $\rho \propto M/R^3$ and $P \propto GM^2/R^4$ in (5.1) reveals the dependence of temperature on mass and radius of the Sun $T \propto \mu M/R$, where the constant of proportionality depends on the mass distribution and the radial distance. We obtain the detailed

temperature distribution throughout the Sun by using equations (2.1) and (4.2) to rewrite the equation of state given in (5.1), that is:

$$
\begin{aligned}
T(\frac{r}{R_\odot}) \;=\;& 3\frac{\mu}{kN_A}G\frac{M_\odot}{R_\odot}\left[\frac{(\frac{3}{\delta}+1)(\frac{3}{\delta}+2)\dots(\frac{3}{\delta}+\gamma)}{\gamma!}\right] \\
& \times\frac{1}{\delta^2}\frac{1}{[1-(\frac{r}{R_\odot})^\delta]^\gamma}\sum_{m=o}^{\infty}\frac{(-\gamma)_m}{m!(\frac{3}{\delta}+m)(\frac{2}{\delta}+m)}\left[\frac{\gamma!}{(\frac{2}{\delta}+m+1)_\gamma}\right. \\
& \left. -\left(\frac{r}{R_\odot}\right)^{m\delta+2}{}_2F_1\left(-\gamma,\frac{2}{\delta}+m;\frac{2}{\delta}+m+1;\left(\frac{r}{R_\odot}\right)^\delta\right)\right],
\end{aligned}
$$

(5.2)

where $_2F_1$ is Gauss' hypergeometric function. Equation (5.2) satisfies the boundary condition $T(r = R_\odot) = 0$ and allows to determine the central value of temperature of the Sun as a function of δ and γ contained in equation (2.1):

$$
\begin{aligned}
T_c \;=\;& 3\frac{\mu}{kN_A}G\frac{M_\odot}{R_\odot}\left[\frac{(\frac{3}{\delta}+1)(\frac{3}{\delta}+2)\dots(\frac{3}{\delta}+\gamma)}{\gamma!}\right] \\
& \times\frac{1}{\delta^2}\sum_{m=0}^{\infty}\frac{(-\gamma)_m\gamma!}{m!(\frac{3}{\delta}+m)(\frac{2}{\delta}+m)(\frac{2}{\delta}+m+1)_\gamma}.
\end{aligned}
$$

(5.3)

At this point of the procedure to construct a model for the internal structure of the Sun by assuming the matter density distribution and subsequently integrating the system of differential equations, two remarks are in place. The contributions of the radiation pressure to the total pressure and the radial dependence of the mean modecular weigth μ have been neglected in equation (5.1).

The total pressure P at any point in the Sun is the sum of the gas pressure and the radiation pressure, $P = P_g + P_r$, where P_g is given in equation (5.1) and $P_r = \frac{1}{3}aT^4$, where a is a constant. Writing $P_g = \beta P$ and hence $P_r = (1 - \beta)P$, it follows that $P = aT^4/3(1-\beta)$. This ratio of radiation pressure to gas pressure

increases towards the center of the Sun but even there the gas pressure exceeds the radiation pressure by more than three orders of magnitude. This justifies that the radiation pressure has been neglected in equation (5.1) (Chandrasekhar, 1939).

The outward flow of energy in the interior of the Sun is driven by the temperature gradient and resisted by the opacity of the material. The nuclear energy generated within the Sun has to continually replenish that radiated away from the surface. This energy generation by nuclear reactions causes the solar chemical composition to change and keeps the Sun evolving. Since the gas in the solar interior is completely ionised, the mean molecular weight μ in equation (5.1) is given by $\mu = (2X + \frac{3}{4}Y + \frac{1}{2}Z)^{-1}$, where X, Y, Z are relative abundances by mass of hydrogen, helium, and heavy elements $(X + Y + Z = 1)$. The dependence of X, Y, Z on the radial distance variable r, which is governed by kinetic equations, can not be determined in the procedure of constructing an analytic solar model by assuming a matter density distribution as given in equation (2.1). Thus, the mean molecular weigth has to be treated as constant in the following. This assumption does not reflect the situation in the real Sun because nuclear reactions have changed the originally uniform chemical composition throughout the Sun.

6 Nuclear Energy Generation Rate

Nuclear energy production in the Sun depends steeply on the temperature of the material and is very concentrated towards the center of the Sun. This is one of the major reasons that calculations of the internal structure of solar-type-stars made already considerable progress before the physical mechanism of the production of energy by nuclear reactions was understood. The rate of nuclear energy generation can be written (Mathai and Haubold, 1988),

$$(6.1) \qquad \epsilon(\rho, T) = \epsilon_0 \rho^n(r) T^m(r),$$

where ϵ_0 is a physical constant depending only on the chemical composition of the solar material and the units chosen. Substituting $\rho(r)$ and $T(r)$ in equation (6.1) by equations (2.1) and (5.2), respectively, and taking advantage of the fact that $0 \leq (\frac{r}{R_\odot}) \leq 1$, the energy generation rate in equation (6.1) can be represented in the form of a polynomial

$$(6.2) \qquad \epsilon\left(\frac{r}{R_\odot}\right) = \epsilon_0 \rho_c^n T_c^m f\left(\frac{r}{R_\odot}\right)^{\delta s + 2q + \delta[n_1 + 2n_2 + \ldots + (2\gamma)n_{2\gamma}]},$$

where f denotes the expression

$$f(\delta, \gamma, m, n; s, q, n_0, n_1, \ldots, n_{2\gamma}; a_0, a_1, \ldots, a_{2\gamma})$$

$$= \sum_{s=0}^{\gamma(m-n)} \frac{[\gamma(m-n)]_s}{s!} \sum_{q=0}^{m} \frac{(-m)_q}{q!} \left(\frac{1}{\eta(\gamma)}\right)^q$$

$$(6.3) \qquad \times \sum_{n_0=0}^{q} \ldots \sum_{n_{2\gamma}=0}^{q} \frac{q! a_0^{n_0} a_1^{n_1} \ldots a_{2\gamma}^{n_{2\gamma}}}{n_0! n_1! \ldots, n_{2\gamma}!},$$

$$n_0 + n_1 + \ldots + n_{2\gamma} = q,$$

and

$$(6.4) \qquad \eta(\gamma) = \sum_{\nu=0}^{\gamma} \frac{(-\gamma)_\nu}{\nu!} \frac{1}{(\frac{2}{\delta}+\nu)(\frac{3}{\delta}+\nu)} \frac{\gamma!}{(\frac{2}{\delta}+\nu+1)_\gamma}.$$

Note that the representation of $\epsilon(r/R_\odot)$ in equation (6.2) is essentially determined by the four free parameters δ, γ, n and m in equation (2.1) and (6.1) with the only restriction that γ be a positive integer. The coefficients $a_0, a_1, \ldots a_{2\gamma}$ in equation (6.3) are determined by the following polynomial of degree 2γ in $\left(\frac{r}{R_\odot}\right)^\delta$:

$$
\begin{aligned}
&\sum_{m_1=0}^{\gamma} \sum_{m_2=0}^{\gamma} \frac{(-\gamma)_{m_1}}{m_1!} \frac{(-\gamma)_{m_2}}{m_2!} \frac{1}{\left(\frac{2}{\delta}+m_1\right)\left(\frac{3}{\delta}+m_1\right)\left(\frac{2}{\delta}+m_1+m_2\right)} \\
(6.5) \qquad &\times \left[\left(\frac{r}{R_\odot}\right)^\delta\right]^{m_1+m_2} = \sum_{m_3=0}^{2\gamma} a_{m_3} \left[\left(\frac{r}{R_\odot}\right)^\delta\right]^{m_3}.
\end{aligned}
$$

7 Luminosity Function

Let $L(r)$ be the function representing the flow of integrated radiation across a sphere of radius r. If ϵ is the energy produced per unit time by nuclear reactions in each unit mass of solar material, than the balance between energy generation in the solar interior and energy loss through its surface is governed by the equation of energy conservation

$$(7.1) \qquad \frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r),$$

where $\epsilon(r)$ is given by equation (6.2). Integrating equation (7.1) over the Sun's interior leads to the luminosity function in terms of Gauss' hypergeometric func-

tion:

$$L\left(\frac{r}{R_\odot}\right) = 4\pi\epsilon_0\rho_c^{n+1}T_c^m R_\odot^3$$

(7.2)
$$\times\frac{1}{\delta}f\frac{1}{s^*}\left(\frac{r}{R_\odot}\right)^{\delta s^*}{}_2F_1\left(-\gamma,s^*;s^*+1;\left(\frac{r}{R_\odot}\right)^\delta\right),$$

$$s^* = s + \frac{1}{\delta}(3+2q) + n_1 + 2n_2 + \ldots + (2\gamma)n_{2\gamma},$$

where f is given in equation (6.3) with s substituted by s^*. Equation (7.2)

satisfies the condition $L(r = 0) = 0$ and gives for the total energy output

$L(r = R_\odot) = L_\odot$,

(7.3)
$$L_\odot = 4\pi\epsilon_0\rho_c^{n+1}T_c^m R_\odot^3\frac{1}{\delta}f\frac{\gamma!}{s^*(s^*+1)\ldots(s^*+\gamma)}.$$

8 Conclusions

Assuming a two-parameter family of matter density distributions in equa-

tion (2.1) made possible the analytic integration of the differential equations

for conservation of mass, momentum, and energy throughout the Sun, equa-

tions (3.1), (4.1) and (7.1), respectively. This procedure shows that hydrostatic

equilibrium, equation of state, and overall energy conservation determine the

state of the central solar conditions, particularly the gravitationally stabilized

solar fusion reactor. The mathematical method chosen reveals the factors of

proportionality which are kept undetermined in dimensional analysis commonly

pursued to understand astrophysical relationships between global parameters of

the Sun. A common mathematical element of the derived distributions of mass

(3.2), pressure (4.2), temperature (5.2), and luminosity (7.2) throughout the Sun is Gauss' hypergeometric function $_2F_1(\cdot)$ which is numerically easily accessible through programms for doing mathematics by computer like Mathematica (Wolfram, 1993).

It has been emphasized above that the assumption of an analytic matter density distribution means that the equation for the transport of energy by radiation through the Sun can be satisfied at only one specific point in the Sun. The flow of radiant energy per unit area through the Sun is proportional to the ratio of radiation pressure gradient and opacity per unit volume. That is

$$(8.1) \qquad H \propto \frac{d(\frac{1}{3}aT^4)/dr}{\kappa\rho} \propto \frac{T^3 dT/dr}{\kappa\rho},$$

where κ denotes the opacity per mass unit at temperature T and density ρ. Because the energy flowing out through the Sun is transported by radiation, we find for the luminosity L

$$(8.2) \qquad L \propto 4\pi R^2 H \propto \frac{R^2 T^3 dT/dr}{\kappa\rho}.$$

Since for a given solar structure $\rho \propto M/R^3$ and $T \propto M/R$, it follows for L that

$$(8.3) \qquad L \propto \frac{1}{\kappa}M^3.$$

For solar composition, Kramer's power law approximation for the opacity given by

$$(8.4) \qquad \kappa \propto \kappa_0 \rho T^{-7/2},$$

where κ_0 is a physical constant depending on the chemical composition of the solar material and the units chosen, which leads to a luminosity -mass- radius

relation,

$$(8.5) \qquad\qquad L \propto M^{11/2} R^{-1/2}.$$

The differential equation governing the outward flow of energy driven by the temperature gradient and resisted by opacity in (8.1) is

$$(8.6) \qquad\qquad L = 4\pi r^2 H = -\left(\frac{16\pi ac}{3\kappa\rho}\right) r^2 T^3 \frac{dT}{dr},$$

where c denotes the velocity of light. To satisfy the radiative energy transport equation (8.6), taking into account the density distribution assumed in equation (2.1), the temperature gradient dT/dr in equation (8.6) has to be equal to the temperature gradient in equation (5.2). This condition can be satisfied at only one specific point in the solar interior, for example at the boundary of the nuclear energy producing core region (at $r \approx 0.3 R_\odot$ where $L \approx L_\odot$).

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